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Homotopy perturbation method for nonlinear MHD Jeffery–Hamel problem

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ABSTRACT

The article solves the Jeffery–Hamel flow using the homotopy perturbation method, an explicit analytical solution is obtained, and the effect of external magnetic field is studied. Comparison of the obtained result with the numerical one reveals validity of the used method.

1. Introduction

The problem of an incompressible, viscous fluid between non-parallel walls, commonly known as the Jeffery–Hamel flow is one of the most applicable cases in fluid mechanics, civil, environmental, mechanical and bio-mechanical engineering. The mathematical investigations of this problem were pioneered by Jeffery [1] and Hamel [2]. Jeffery–Hamel flows are an exact similarity solution of the Navier–Stokes equations in the special case of two-dimensional flow through a channel with inclined plane walls meeting at a vertex, and with a source or sink at the vertex. The issue has been extensively studied by several authors and discussed in many textbooks [3–5]. The classical Jeffery–Hamel problem was extended in [6] to include the effects of external magnetic field on conducted fluid. The magnetic field acts as a control parameter, along with the flow Reynolds number and the angle of the walls. In the MHD Jeffery–Hamel problem, we have to deal with a nonlinear system.

Most scientific problems and phenomena such as heat transfer and fluid flow occur nonlinearly. Except for a limited number of these problems, in most cases it is difficult to find the exact analytical solutions for them. Therefore, approximate analytical solutions have been sought and introduced among which HPM is the most effective and convenient ones for both weakly and strongly nonlinear equations. This method was first introduced by He [7]. Because this method continuously deforms a difficult problem into a simple one, which is easy to solve, this method has been used by many authors such as Ganji [8] and the references therein to handle a wide variety of scientific and engineering applications such as linear and nonlinear, homogeneous equations as well. In this case study, we have applied He's HPM to find the approximate solutions of nonlinear differential equations governing the MHD Jeffery–Hamel flow, and a comparison between the results and the numerical solution has been provided.

2. Problem statement and governing equation

Consider a system of cylindrical polar coordinates (r, θ, z) with a steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at channel walls lying in planes, and intersecting in the axis of z (Fig. 1).

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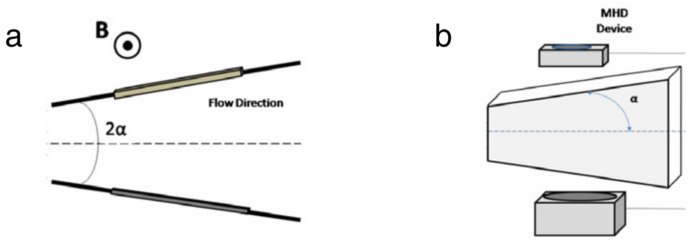

Fig. 1. Geometry of the MHD Jeffery–Hamel flow; (a) a 2-D view and (b) Schematic setup of problem.

The governing equations are [9]:

$$\frac{\rho\partial}{r\partial r}(ru(r,\theta)) = 0 \tag{1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \upsilon\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right] - \frac{\sigma B_0^2}{\rho r^2}u(r,\theta) \tag{2}$$

$$\frac{1}{\rho r}\frac{\partial P}{\partial \theta} - \frac{2\upsilon}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0 \tag{3}$$

where B_0, σ, $u(r,\theta)$, P, υ, ρ are the electromagnetic induction, the conductivity of the fluid, the velocity along radial direction, the fluid pressure, the coefficient of kinematic viscosity and the fluid density, respectively. From Eq. (1) and using dimensionless parameters we get:

$$f(\theta) = ru(r,\theta) \tag{4}$$

$$f(\eta) = \frac{f(\theta)}{f_{\max}}, \quad \eta = \frac{\theta}{\alpha}. \tag{5}$$

Substituting Eq. (5) into Eqs. (2) and (3) and eliminating P, we obtain an ordinary differential equation for the normalized function profile $f(\eta)$ [8]:

$$f'''(\eta) + 2\alpha Re f(\eta)f'(\eta) + (4-H)\alpha^2 f'(\eta) = 0. \tag{6}$$

With the boundary conductions:

$$f(0) = 1, \qquad f'(0) = 0, \qquad f(1) = 0. \tag{7}$$

The Reynolds number is:

$$Re = \frac{f_{\max}\alpha}{\upsilon} = \frac{U_{\max} r\alpha}{\upsilon}\begin{pmatrix} divergent - channel: \alpha \succ 0, f_{\max} \succ 0 \\ convergent - channel: \alpha \prec 0, f_{\max} \prec 0 \end{pmatrix}. \tag{8}$$

The Hartmann number is:

$$H = \sqrt{\frac{\sigma B_0^2}{\rho\upsilon}}. \tag{9}$$

3. Application of HPM on MHD Jeffry–Hamel flow

We define *He's* general form of Eq. (6) as:

$$A(F) - U(\eta) = 0, \quad \eta \in \Omega. \tag{10}$$

With the boundary condition of:

$$B\left(F, \frac{\partial F}{\partial n}\right) = 0, \quad \eta \in \Gamma \tag{11}$$

where A is a general differential operator, B a boundary operator, $U(\eta)$ a known analytical function and Γ is the boundary of the domainΩ.

So according to Eq. (6) we will have:

$$A(F) = \frac{\partial^3 F(\eta)}{\partial \eta^3} + 2\alpha Re F(\eta)\frac{\partial F(\eta)}{\partial \eta} + (4-H)\alpha^2\frac{\partial F(\eta)}{\partial \eta}, \qquad U(\eta) = 0. \tag{12}$$

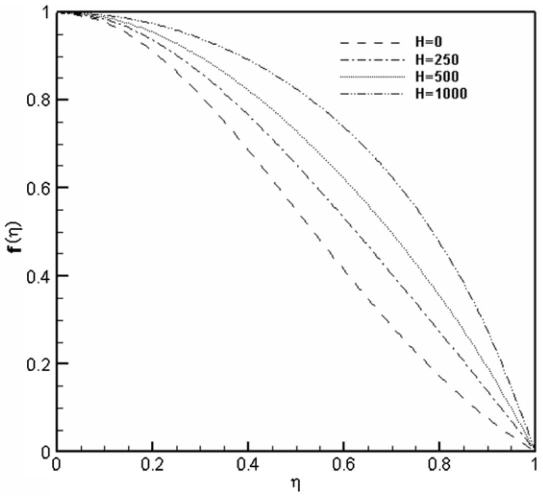

Fig. 2. Velocity profile for $\alpha = 7.5^\circ$, $Re = 50$ and various Hartmann number.

A can be divided into two parts, L and N, where L is linear and N is nonlinear:

$$L(F) + N(F) = 0, \quad \eta \in \Omega. \tag{13}$$

The homotopy perturbation structure is shown as follows:

$$H(F,p) = (1-p)[L(F) - L(f_0)] + p[A(F) - U(r)] = 0 \tag{14}$$

where, $p \in [0, 1]$ is an embedding parameter and f_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (6) can be written as a power series in p, as following:

$$F(\eta) = F_0(\eta) + pF_1(\eta) + p^2F_2(\eta) + \cdots = \sum_{n=0}^{\infty} p^n F_n(\eta). \tag{15}$$

Considering Eqs. (14) and (6), we will have:

$$(1-p)(F'''(\eta) - f'''(\eta_0)) + p(F'''(\eta) + 2\alpha ReF(\eta)F'(\eta) + (4-H)\alpha^2 F'(\eta)) = 0. \tag{16}$$

By substituting $F(\eta)$ from Eq. (15) into Eq. (16) and after some simplifications and rearrangements based on powers of p-terms, we have:

$$p^0: \quad \frac{\partial^3 F_0(\eta)}{\partial \eta^3} = 0 \quad \text{with boundary conditions } F_0(0) = 1,\ F_0'(0) = 0,\ F_0''(0) = a. \tag{17}$$

$$p^1: \quad \frac{\partial^3 F_1(\eta)}{\partial \eta^3} + 2\alpha ReF_0(\eta)\left(\frac{\partial F_0(\eta)}{\partial \eta}\right) + \alpha^2(4-H)\left(\frac{\partial F_0(\eta)}{\partial \eta}\right) = 0. \tag{18}$$

Boundary conditions are $F_1(0) = 0, F_1'(0) = 0, F_1''(0) = 0$.

$$p^2: \quad \frac{\partial^3 F_2(\eta)}{\partial \eta^3} + \alpha^2(4-H)\left(\frac{\partial F_1(\eta)}{\partial \eta}\right) + 2\alpha ReF_0(\eta)\left(\frac{\partial F_1(\eta)}{\partial \eta}\right) + 2\alpha ReF_1(\eta)\left(\frac{\partial F_0(\eta)}{\partial \eta}\right) = 0. \tag{19}$$

Boundary conditions are $F_0(0) = 0, F_2'(0) = 0, F_2''(0) = 0$.
Solving Eq. (17) to (19) with boundary conditions, we have:

$$F_0(\eta) = \frac{1}{2}a\eta^2 + 1 \tag{20}$$

$$F_1(\eta) = -\frac{1}{120}aR\alpha^2\eta^6 + \left(-\frac{1}{6}a\alpha^2 + \frac{1}{24}aH\alpha^2 - \frac{1}{12}aR\alpha\right)\eta^4 \tag{21}$$

$$\begin{aligned} F_2(\eta) = {} & \frac{1}{10800}a^3R^2\alpha^2\eta^{10} + \left(\frac{1}{280}a^2R\alpha^3 + \frac{1}{560}a^2R^2\alpha^2 - \frac{1}{1120}a^2HR\alpha^3\right)\eta^8 \\ & + \left(-\frac{1}{90}aH\alpha^4 + \frac{1}{720}aH^2\alpha^4 - \frac{1}{180}aRH\alpha^3 + \frac{1}{45}a\alpha^4 + \frac{1}{45}aR\alpha^3 + \frac{1}{180}aR^2\alpha^2\right)\eta^6. \end{aligned} \tag{22}$$

Table 1
The comparison between the numerical and HPM results for f (velocity) in divergence channel when: $\alpha = 7.5°$, $Re = 50$.

η	$H = 0$			$H = 250$			$H = 500$			$H = 1000$		
	HPM	*Nu*	*Error*	*HPM*	*Nu*	*Error*	*HPM*	*Nu*	*Error*	*HPM*	*Nu*	*Error*
0.0	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000	0.0000000
0.1	0.9770711	0.9771426	0.0000715	0.9837340	0.9837368	0.0000028	0.9883197	0.9883197	0.0000000	0.9937607	0.9937611	0.0000004
0.2	0.9112020	0.9114792	0.0002772	0.9363350	0.9363460	0.0000110	0.9537955	0.9537953	0.0000002	0.9747886	0.9747905	0.0000019
0.3	0.8104115	0.8110052	0.0005937	0.8616894	0.8617134	0.0000240	0.8978515	0.8978511	0.0000004	0.9422794	0.9422839	0.0000045
0.4	0.6859230	0.6869148	0.0009918	0.7653405	0.7653815	0.0000410	0.8224699	0.8224691	0.0000008	0.8947431	0.8947514	0.0000083
0.5	0.5498427	0.5512883	0.0024456	0.6533961	0.6534573	0.0000612	0.7296817	0.7296804	0.0000013	0.8297471	0.8297606	0.0000135
0.6	0.4131698	0.4151089	0.0019391	0.5314621	0.5315467	0.0000846	0.6209748	0.6209729	0.0000019	0.7434756	0.7434960	0.0000204
0.7	0.2846024	0.2870546	0.0015478	0.4038130	0.4039228	0.0001098	0.4966644	0.4966618	0.0000026	0.6299866	0.6300164	0.0000298
0.8	0.1702791	0.1731221	0.0028430	0.2728708	0.2729980	0.0001272	0.3552115	0.3552079	0.0000036	0.4799338	0.4799758	0.0000420
0.9	0.0744232	0.0768716	0.0024484	0.1389433	0.1390416	0.0000938	0.1923821	0.1923775	0.0000046	0.2782789	0.2783299	0.0000510
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

And the best approximation for the solution following Eq. (15) is:

$$f(\eta) = \lim_{p\to 1} F(\eta) = F_0(\eta) + F_1(\eta) + F_2(\eta) + \cdots. \tag{23}$$

Solutions $F_3(\eta)$ to $F_n(\eta)$ were too long to be mentioned here. Herein the parameter a can be determined by boundary condition $F(1) = F_0(1) + F_1(1) + F_2(1) + \cdots = 0$ for $\alpha = 7.5°$, $R = 50$ and various values of H. The numerical results are obtained using the inbuilt maple software (boundary value problem solver). Fig. 2 indicates that increasing the Hartmann number leads to higher velocity which has a great effect on the performance of the system. Also, the comparison between the numerical and HPM results for the velocity profile in the divergence channel when $\alpha = 7.5°$, $Re = 50$ is shown at Table 1. The solution shows that the results of the present method are in excellent agreement with those of the numerical ones.

4. Conclusions

In this paper, the homotopy perturbation method has been successfully applied to a nonlinear MHD Jeffery–Hamel problem. It could be noticed that by increasing H, the velocity profile increases resulting in a rise in the flow rate. It is also shown that the results obtained by the HPM method are in acceptable agreement with ones derived by the numerical method.

References

[1] G.B. Jeffery, The two-dimensional steady motion of a viscous fluid, Phil. Mag. 6 (1915) 455–465.
[2] G. Hamel, Spiralförmige Bewgungen Zäher Flüssigkeiten, Jahresber. Deutsch. Math. -Verein. 25 (1916) 34–60.
[3] L. Rosenhead, The steady two-dimensional radial flow of viscous fluid between two inclined plane walls, Proc. R. Soc. A 175 (1940) 436–467.
[4] K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.
[5] Reza M. Sadri, Channel entrance flow, Ph.D. Thesis, Department of Mechanical Engineering, The University of Western Ontario, 1997.
[6] W.I. Axford, The Magnetohydrodynamic Jeffrey–Hamel problem for a Weakly conducting fluid, Q. J. Mech. Appl. Math. 14 (1961) 335–351.
[7] J.H. He, Homotopy perturbation technique, J. Comput. Methods Appl. Mech. Engrg. 17 (8) (1999) 257–262.
[8] D.D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Phys. Lett. A 355 (2006) 337–341.
[9] Hermann Schlichting, Boundary-Layer Theory, McGraw-Hill Press, New York, 2000.